On the Combination of Quality Indicators for Multi-Objective Optimization

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***Abstract*—For almost two decades, quality indicators (QIs) have been used to assess and compare Multi-Objective Evolutionary Algorithms (MOEAs). Each QI represents different preferences of the Decision Maker which implies that not all indicators rank the approximation sets generated by MOEAs in a similar fashion. In fact, they are sometimes in conflict. The most general property that a QI should have is Pareto-compliance, which means that every time an approximation set is better than another in a Pareto sense, the indicator must reflect this. Regarding unary indicators, the hypervolume is the only known Pareto-compliant indicator, and the remaining are weakly Pareto-compliant or non Pareto-compliant. Additionally, to the authors' best knowledge, there is not theoretical study that considers the combination of indicators in order to produce new ones. In this paper, we propose a theoretical basis to combine existing weakly Pareto-compliant indicators with at least one Pareto-compliant, being the resulting combined indicator Pareto-compliant. The consequences of these new combined indicators are threefold: 1) increase the variety of available indicators to achieve/adjust desired distributions on the Pareto front, 2) correct weakly Pareto-compliant indicators such that they are Pareto-compliant, and 3) generate new selection mechanisms for MOEAs.**

***Index Terms*—Quality Indicators, Combined Quality Indicators, Multi-Objective Optimization, Indicator-based selection**

I. Introduction

EVOLUTIONARY Multi-Objective Optimization (EMOO) originated in the mid-1980s, has been steadily growing since the late 1990s, focusing on the solution of problems with several, often conflicting, objective functions [1]. These problems are the so-called multi-objective optimization problems (MOPs). Due to the conflict among the objectives, the solution to a MOP consists of an infinite set of vectors, denoted as the *Pareto optimal front*, that represents the best possible trade-offs among them.

Through the years, different techniques have been developed to solve MOPs [1], [2]. However, Multi-Objective Evolutionary Algorithms (MOEAs) have arose as a popular option in the last three decades for solving highly complex MOPs. MOEAs are stochastic set-based metaheuristics inspired by the principles of natural evolution of organisms. Commonly, MOEAs have employed the Pareto dominance relation[1] in furtherance of selecting the fittest solutions. MOEAs produces at each execution a discrete set of solutions that aim to approximate the Pareto optimal front. Depending on the design of each MOEA, the approximation sets will exhibit different properties. According to Zitzler *et al.* [3], the most representative features of an approximation set are: 1) the distance between it and the Pareto optimal front, denoted as convergence, 2) the uniform distribution of points, and 3) the spread of solutions. Figure 1 shows various approximation sets, having the above mentioned characteristics, where case IV is the ideal one because it exhibits a convergent approximation that covers the entire Pareto optimal front and it is evenly distributed.

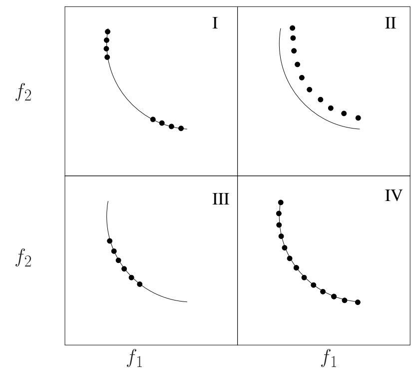

Fig. 1. Different cases of convergence, distribution and spread of solutions in Pareto fronts. The continuous line depicts the Pareto optimal front and the black points represent an approximation set.

Currently, there is a wide variety of MOEAs in the specialized literature [4], [5], [6], [7], [8], [9]. Due to the No-Free Lunch Theorem, a MOEA cannot show a good performance in all types of MOPs. Hence, an important question is to determine which is the best MOEA for each MOP. At first, researchers visually compared the approximated Pareto fronts generated by MOEAs [10]. It was until the mid-1990s that some isolated efforts were undertaken to try to (numerically) assess the performance of MOEAs [11], [12], [13] by using quality indicators (QIs) which are functions that assign a real value to an approximated Pareto front. However, the PhD thesis of David Van Veldhuizen [14] can be considered as the cornerstone of QIs due to his comprenhensive review of most of the currently available indicators. Moreover, due to to stochastic nature of MOEAs, Van Veldhuizen and Lamont [15] proposed a non-parametric statistical method to analyze their performance based on QIs values. Since then, this methodol-

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[1] A vector Pareto dominates another one if the former is as good as the latter in all of the components but better in at least one of them.

ogy has been typically adopted by EMOO community in order to draw conclusions.

Mathematically, QIs are functions that assign a real value to an approximation set, i.e., they impose a total order among sets of solutions [16]. However, the overall complexity of QIs is greater because they posses several properties that are shown in Fig. 2. The number of approximation sets that a QI is able to assess determines its cardinality. The performance criteria is related to what a QI measures: capacity, convergence, diversity (divided into distribution and spread of solutions) or convergence-diversity. Pareto compliance is directly related to convergence QIs and it roughly states if a QI is compliant with the Pareto dominance relation. The sensitivity of a QI to the different units and scales of the objective functions determines its scaling invariance. If the indicator computation requires knowledge of the MOP being solved, then it is knowledge-dependent. Similarly, a QI is parameter-dependent if it needs user-supplied parameters. A critical aspect in the practical field is its runtime complexity. Additionally, scalability in objective space is an important characteristic that should be taken into account as current MOEAs are generating approximation sets in highdimensional objective spaces, thus, it is crucial to know if the indicator correctly evalutes them.

Since the late 1990s, several QIs have been proposed to assess different aspects of the Pareto front approximations [17], [18]. However, convergence QIs are the most important, being the hypervolume (HV) indicator the most representative [19]. HV is a convergence-diversity QIs that measures the dominated volume between the approximation set and an anti-optimal reference point. Additionally, HV is the only unary indicator that is known to be Pareto-compliant [20]. Despite its nice mathematical properties, HV is highly computational expensive when the dimension of the objective space increases. Consequently, other QIs have been proposed such as the R2 [21], the Inverted Generational Distance plus indicator (IGD$^+$) [22] and the ϵ^+ indicator [16]. These QIs are computationally cheaper than HV but they have weaker mathematical properties, i.e., they are weakly Pareto-compliant[2]. However, there is not a single QI that assesses all the desired features of an approximation set, since each indicator shows different properties [16], [23].

In this paper, we introduce a mathematical framework to combine unary Pareto-compliant and weakly Pareto-compliant indicators by using order-preserving utility functions. Hence, a new family of Pareto-compliant utility indicators (PCUIs) is defined. The purpose of these PCUIs are twofold: 1) increase the number of Pareto-compliant indicators having different preferences, and 2) correct weakly Pareto-compliant by making them Pareto-compliant. Since currently the only unary Pareto-compliant indicator is the hypervolume, we use it as the base to form PCUIs and, additionally, we employ the R2, IGD$^+$, and ϵ^+ indicators that are weakly Pareto-compliant. In order to prove the usefulness of PCUIs, we analyze their preferences by correlating the way they rank various distribution of points generated by state-of-the-art MOEAs.

[2]See. Property 2.

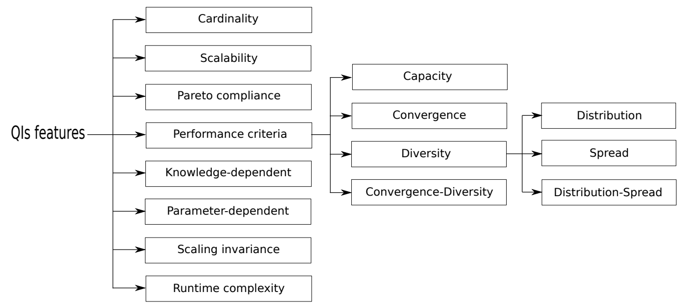

Fig. 2. Main features of quality indicators.

The remainder of the paper is organized as follows. Section II outlines the related work. A set of mathematical concepts required for the understanding of the paper are defined in Sect. III. The mathematical framework of the combination of QIs is introduced in Sect. IV. Section V presents the experimental results using the combined indicators. Finally, the main conclusions and future work are described in Sect. VI.

II. Previous Related Work

To the authors best knowledge, there is no work that proposes the mathematical combination of quality indicators in order to define new ones. However, there is a number of approaches that propose MOEAs using multiple indicators in furtherance of guiding the selection process. This kind of MOEAs combine the individual effect, or bias, of each indicator in order to select solutions and drive the population to the Pareto optimal front. In the following, these proposals are introduced.

The core idea of Multi-Indicator-based MOEAs (MIB-MOEAs) is to combine the properties of each indicator-based mechanism in order to obtain a global search behavior. Phan and Suzuki [24] were apparently the first to propose a MIB-MOEA (called BIBEA) that boosts existing IB-selection operators, using the AdaBoost algorithm. The proposed multi-indicator selection aims to select the potential parents for crossover. In a further work, Phan *et al.* [25] improved BIBEA's parent selection by using an ensemble learning method and they also proposed a multi-indicator environmental selection. An issue of both proposals is that they required supervised off-line training, using certain MOPs. Hence, apparently, they could not solve any MOP. Unfortunately, the experimental results did not show that the proposals outperformed state-of-the-art MOEAs.

On the other hand, Hernández and Coello [26] proposed a MOEA, called Many-Objective Metaheuristic Based on the R2 Indicator (MOMBI-III), that combines the convergence effect of an R2-selection mechanism and a density estimator based on the s-energy indicator [27] for improving diversity. Additionally, the $R2$-selection employs a hyper-heuristic to select the most suitable utility function for the $R2$ indicator. Their experimental results showed that MOMBI-III outperforms state-of-the-art MOEAs.

Finally, in 2018, Falcón-Cardona and Coello [6] proposed the Multi-Indicator Evolutionary Hyper-Heuristic (MIHPS) that has a pool of four selection density estimators based on the R2, IGD$^+$, ϵ^+ and Δ_p [28] indicators. MIHPS analyzes the convergence behavior of each indicator-based density estimator (IB-DE) in order to update the transition weights of a Markov chain. Hence, the core idea is to select under certain probability the best IB-DE depending on the online behavior of the algorithm. The main result of MIHPS is that the IB-DEs based on IGD$^+$, ϵ^+ and Δ_p improve the convergence behavior and are suitable for the early stages of the evolutionary process, meanwhile the R2 indicator is better to improve diversity and refine convergence at the end of the evolutionary process.

III. Background

In this section, we provide the mathematical basis for the understanding of the paper. First, we briefly introduce the basic concepts related to multi-objective optimization, following the definitions provided by Coello *et al.* [1]. Then, we review four state-of-the-art quality indicators (namely, HV, R2, IGD$^+$ and ϵ^+) because of their importance in the EMOO community.

A. Multi-Objective Optimization

The general multi-objective optimization problem[3] (MOP) is formulated as:

$$\min_{\vec{x}\in\mathbb{R}^n} \vec{F}(\vec{x}) := [f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x})]^T \quad (1)$$

subject to:

$$g_i(\vec{x}) \leq 0 \quad i = 1, 2, \ldots, k \quad (2)$$

$$h_j(\vec{x}) = 0 \quad j = 1, 2, \ldots, p \quad (3)$$

where $\vec{x} = (x_1, x_2, \ldots, x_n)^T$ is the n-dimensional vector of decision variables; $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are the objective functions and $g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, $j = 1, \ldots, p$ are the constraint functions of the problem which define that feasible region Ω.

Definition 1 (Weak Pareto Dominance)**.** *Given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, we say that $\vec{u}$* **weakly dominates** *$\vec{v}$ (denoted by $\vec{u} \preceq \vec{v}$) if $u_i \leq v_i$ for all $i = 1, \ldots, n$.*

Definition 2 (Pareto Dominance)**.** *Given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, we say that $\vec{u}$* **dominates** *$\vec{v}$ (denoted by $\vec{u} \prec \vec{v}$) if $u_i \leq v_i$ for $i = 1, \ldots, n$ and there exists at least an index $j \in \{1, \ldots, m\}$ such that $u_j < v_j$.*

Definition 3 (Strict Pareto Dominance)**.** *Given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, we say that $\vec{u}$* **strictly dominates** *$\vec{v}$ (denoted by $\vec{u} \prec\prec \vec{v}$) if $u_i < v_i$ for all $i = 1, \ldots, n$.*

Definition 4 (Pareto Optimality)**.** *We say that a vector of decision variables $\vec{x}^* \in \Omega$ is* **Pareto optimal** *if there does not exist another $\vec{x} \in \Omega$ such that $\vec{F}(\vec{x}) \prec \vec{F}(\vec{x^*})$.*

[3]Without loss of generality, we will assume only unconstrained minimization problems. To transform a minimization problem into a maximization one, we can use: $\max f = -\min(-f)$

TABLE I
Relations on approximation sets based on Pareto dominance relations. $\mathcal{A} \prec\prec \mathcal{B} \Rightarrow \mathcal{A} \prec \mathcal{B} \Rightarrow \mathcal{A} \lhd \mathcal{B} \Rightarrow \mathcal{A} \preceq \mathcal{B}$.

Relation	Description
$\mathcal{A} \prec\prec \mathcal{B}$	$\forall \vec{b} \in \mathcal{B}, \exists \vec{a} \in \mathcal{A} : \vec{a} \prec\prec \vec{b}$
$\mathcal{A} \prec \mathcal{B}$	$\forall \vec{b} \in \mathcal{B}, \exists \vec{a} \in \mathcal{A} : \vec{a} \prec \vec{b}$
$\mathcal{A} \lhd \mathcal{B}$	$\forall \vec{b} \in \mathcal{B}, \exists \vec{a} \in \mathcal{A} : \vec{a} \preceq \vec{b} \wedge \mathcal{A} \neq \mathcal{B}$
$\mathcal{A} \preceq \mathcal{B}$	$\forall \vec{b} \in \mathcal{B}, \exists \vec{a} \in \mathcal{A} : \vec{a} \preceq \vec{b}$
$\mathcal{A} \parallel \mathcal{B}$	$\mathcal{A} \npreceq \mathcal{B} \wedge \mathcal{B} \npreceq \mathcal{A}$

Definition 5. *The* **Pareto Optimal Set** *$\mathcal{P}^*$ is defined by:*

$$\mathcal{P}^* = \{\vec{x}^* \in \Omega \mid \vec{x} \text{ is Pareto optimal}\}$$

Definition 6. *The* **Pareto Front** *$\mathcal{PF}^*$ is defined by:*

$$\mathcal{PF}^* = \{\vec{F}(\vec{x}^*) \in \mathbb{R}^m \mid \vec{x}^* \in \mathcal{P}^*\}$$

Definition 7 (Approximation Set)**.** *Let $\mathcal{A} \in \Psi$ be a finite set of m-dimensional objective vectors. $\mathcal{A}$ is called an* ***approximation set*** *or* ***approximate Pareto front*** *if $\forall \vec{u}, \vec{v} \in \mathcal{A}, \vec{u} \neq \vec{v}$ it holds that $\vec{u} \npreceq \vec{v} \wedge \vec{v} \npreceq \vec{u}$. The set of all approximation sets is denoted as Ψ.*

Definition 8. *The Ideal Objective Vector ($\vec{z}^* \in \mathbb{R}^m$) is constructed using the minimum of each of the objective functions, considered separately. Each i^{th}-component of the ideal vector is defined as $z_i^* = \min_{\vec{x}} f_i(\vec{x})$.*

Definition 9. *The Nadir Objective Vector ($\vec{z}^{nad} \in \mathbb{R}^m$) is constructed using the worst values of $\mathcal{PF}^*$. Each i^{th}-component is defined as $z_i^{nad} = \max_{\vec{x}\in\mathcal{P}^*} f_i(\vec{x})$.*

Based on the three types of Pareto dominance relations, it is possible to define dominance relations between approximation sets, according to Zitzler *et al.* [16]. Table I shows five types of set dominance relations.

B. Review of quality indicators

In this section, we first introduce in Definition 10 what is a unary quality indicator. Then, we mathematically define the indicators: HV, $R2$, IGD$^+$, and ϵ^+. The properties of these indicators are outlined in Table II. In all cases, let $\mathcal{A}$ be an approximation set, $\mathcal{Z}$ be a reference set and m be the dimension of the objective space.

Definition 10 (Unary Quality Indicator)**.** *A unary quality indicator I is a function $I : \Psi \to \mathbb{R}$, which assigns a real value to each approximation set $\mathcal{A}$.*

Definition 11 (Hypervolume)**.** *Let Λ denote the Lebesgue measure, then HV is defined as follows:*

$$HV(\mathcal{A}, \vec{z}_{ref}) = \Lambda\left(\bigcup_{\vec{a}\in\mathcal{A}} \{\vec{x} \mid \vec{a} \prec \vec{x} \prec \vec{z}_{ref}\}\right), \quad (4)$$

TABLE II

PROPERTIES OF THE MOST COMMONLY QIS EMPLOYED BY IB-MOEAS. C MEANS CONVERGENCE AND C/D MEANS CONVERGENCE-DIVERSITY. N DENOTES THE POPULATION SIZE AND M IS RELATED TO THE SIZE OF THE REFERENCE SET OR SET OF WEIGHT VECTORS. m IS THE NUMBER OF OBJECTIVE FUNCTIONS.

Indicator	Cardinality	Performance criteria	Pareto compliance	Knowledge dependent	Parameter dependent	Scaling invariant	Scalable	Runtime complexity	Reference
HV	Unary	C/D	Strict	Reference point	No	No	Yes	High	[19]
R2	Unary	C/D	Weak	Reference point	Weight vectors Utility functions	No	Yes	$\mathcal{O}(mNM)$	[21]
IGD$^+$	Unary	C/D	Weak	Reference set	No	Yes	Yes	$\mathcal{O}(mNM)$	[22]
ϵ^+	Unary	C	Weak	Reference set	No	No	Yes	$\mathcal{O}(mNM)$	[16]

where $\vec{z}_{ref} \in \mathbb{R}^m$ is a reference point which should be dominated by all points in $\mathcal{A}$.

Definition 12 (Unary $R2$ indicator)**.** *The unary $R2$ indicator is defined as follows:*

$$R2(\mathcal{A}, W) = -\frac{1}{|W|}\sum_{\vec{w}\in W}\max_{\vec{a}\in\mathcal{A}}\{u_{\vec{w}}(a)\}, \qquad (5)$$

where W is a set of m-dimensional weight vectors and $u_{\vec{w}} : \mathbb{R}^m \mapsto \mathbb{R}$ is a scalarizing function, parameterized by $\vec{w} \in W$, that assigns a real value to each solution vector.

Definition 13 (Inverted Generational Distance plus)**.** *The IGD$^+$, for minimization, is defined as follows:*

$$IGD^+(\mathcal{A}, \mathcal{Z}) = \frac{1}{|\mathcal{Z}|}\sum_{\vec{z}\in\mathcal{Z}}\min_{\vec{a}\in\mathcal{A}} d^+(\vec{a}, \vec{z}) \qquad (6)$$

where $d^+(\vec{a}, \vec{z}) = \sqrt{\sum_{k=1}^{m}(\max\{a_k - z_k, 0\})^2}$.

Definition 14 (Unary ϵ^+ indicator)**.** *The unary ϵ^+-indicator gives the minimum distance by which a Pareto front approximation needs to or can be translated in each dimension in objective space such that a reference set is weakly dominated. Mathematically, it is defined as follows:*

$$\epsilon^+(\mathcal{A}, \mathcal{Z}) = \max_{\vec{z}\in\mathcal{Z}}\min_{\vec{a}\in\mathcal{A}}\max_{1\le i\le m}\{z_i - a_i\}. \qquad (7)$$

IV. COMBINATION OF QUALITY INDICATORS

For almost two decades, quality indicators have been designed to assess approximation sets generated by multi-objective optimizers [19], [29], [28], [22], [21]. Each indicator represents different preferences and, thus, not all the quality indicators evaluate approximation sets in the same way [23], [30]. In fact, they are sometimes in conflict. However, to the authors' best knowledge, the effect of combining indicators, in order to produce new ones, has not been studied. In this work, we focus on this question and we propose a methodology to combine existing indicators. In the following, we present the theoretical aspects that support the combination of quality indicators.

Definition 15 (Combination function)**.** *A combination function $C : \mathbb{R}^k \to \mathbb{R}$ assigns a real value to a vector $\vec{I} = (I_1, I_2, \ldots, I_k)$, where each I_j is a unary indicator.*

Definition 16 (Combined Indicator)**.** *Given a vector of k indicators $\vec{I} = (I_1, I_2, \ldots, I_k)$ and a combination function C, a combined indicator $\mathcal{I}$ is defined as follows: $\mathcal{I} = C(\vec{I})$.*

Clearly, Definitions 15 and 16 describe a combined indicator $\mathcal{I}$ as a general function that transforms a vector of indicator values into a single real value. However, for getting more important theoretical results, we should say something about the properties of each $I_j, j = 1, \ldots, k$ and the combination function. Hansen and Jaszkiewicz [29] defined when the evaluation of two approximation sets by a certain indicator is compatible with the result of Pareto-based outperformance relation applied to these two sets. Hence, an indicator could be compliant or weakly compliant with the outperformance relation. In our case, let $\lhd$ be the outperformance relation (see Table I). The following two properties formally state both terms. Without loss of generality, let us assume that a greater indicator value corresponds to a higher quality.

Property 1 (Pareto compliance)**.** *Given two approximation sets $\mathcal{A}$ and $\mathcal{B}$, a unary indicator I is $\lhd$-compliant (Pareto compliant) if $\mathcal{A} \lhd \mathcal{B} \Rightarrow I(\mathcal{A}) > I(\mathcal{B})$.*

Property 2 (Weakly Pareto compliance)**.** *Given two approximation sets $\mathcal{A}$ and $\mathcal{B}$, a unary indicator I is weakly $\lhd$-compliant (weakly Pareto compliant) if $\mathcal{A} \lhd \mathcal{B} \Rightarrow I(\mathcal{A}) \ge I(\mathcal{B})$.*

Based on the above definitions, we construct an special vector of indicators that is necessary for the refinement of the combined indicator model.

Definition 17 (Compliant Indicator Vector)**.** *$\vec{I} = (I_1, I_2, \ldots, I_k) \in Q$ is called a compliant indicator vector (CIV) if $\forall j = 1, \ldots, k, I_j$ is weakly Pareto compliant and there exists at least an index $t \in \{1, \ldots, k\}$ such that I_t is Pareto compliant.*

Theorem 1 (Construction of Pareto-compliant combined indicators)**.** *Let $I_1, \ldots, I_k$ be unary indicators that form a compliant indicator vector $\vec{I}$. A combined indicator $\mathcal{I}(\vec{I})$ is $\lhd$-compliant if it has the order-preserving property:*

$$\forall \vec{u}, \vec{v} \in \mathbb{R}^k, \vec{u} \succ \vec{v} \Rightarrow \mathcal{I}(\vec{u}) > \mathcal{I}(\vec{v}).$$

Proof. Consider two approximation sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \lhd \mathcal{B}$ and let $\vec{I}^{\mathcal{A}} = \vec{I}(\mathcal{A})$ and $\vec{I}^{\mathcal{B}} = \vec{I}(\mathcal{B})$, where $\vec{I}$ is a CIV. Then, $\mathcal{A} \lhd \mathcal{B} \Rightarrow \vec{I}^{\mathcal{A}} \succ \vec{I}^{\mathcal{B}}$ because the Pareto-compliant indicators get better and the weakly Pareto-compliant ones get better or stay equal. Moreover, by definition $\vec{I}^{\mathcal{A}} \succ \vec{I}^{\mathcal{B}} \Rightarrow \mathcal{I}(\vec{I}^{\mathcal{A}}) > \mathcal{I}(\vec{I}^{\mathcal{B}})$. Hence, by transitivity of $\Rightarrow$, it holds $\mathcal{A} \lhd \mathcal{B} \Rightarrow \mathcal{I}(\vec{I}^{\mathcal{A}}) > \mathcal{I}(\vec{I}^{\mathcal{B}})$, i.e., $\mathcal{I}$ is Pareto-compliant. □

Theorem 1 provides a sufficient condition for constructing Pareto-compliant combined indicators on the basis of compliant indicator vectors. In other words, a combined indicator preserves the Pareto-compliant property because of the use of order-preserving combination functions.

Remark 1. *The condition of Theorem 1 is suffcient but not necessary. For instance, given $\vec{I} = (I_1, I_2, \ldots, I_k)$ where I_1 is Pareto-compliant and the $I_j, j = 2, \ldots, k$ are not Pareto-compliant, the combined indicator $\mathcal{I}(\vec{I}) = I_1$ is also Pareto-compliant. Hence, there is a big a number of possibilities to construct combined indicators.*

An important question that arises is why it is important to construct new Pareto-compliant indicators. For answering this question, let us consider the Zero indicator (Z) that assigns a zero value to all the approximation sets, i.e., $Z : \Psi \rightarrow 0$. For every $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \prec \mathcal{B}$, it implies $Z(\mathcal{A}) = Z(\mathcal{B})$, i.e., Z weakly Pareto-compliant. Although indicators such as R2, IGD$^+$ and ϵ^+ are more complex than Z in a mathematical sense, all of them are weakly Pareto-compliant. Hence, we can see that is not enough to construct a QI having this property in spite of clear advantage related to have QI less computational expensive than HV. In consequence, there is a need of new Pareto-compliant indicators and weakly Pareto-compliant QIs can be employed to redefine the preferences of the only Pareto-compliant indicator know so far, i.e., the hypervolume.

There exists many combination functions that have the property of Theorem 1. However, in this paper, we focus on certain utility functions [2], [31] $u : \mathbb{R}^k \rightarrow \mathbb{R}$ that hold the desired property. An utility function (UI) is a model of the Decision Maker preferences that assigns to each k-dimensional vector an utility value. Thus, a combination function C can be defined in terms of these functions. Generally, UIs employ a convex weight vector $\vec{w} \in \mathbb{R}^k$ such that $\sum_{i=1}^{k} w_i = 1, w_i \geq 0$. However, for our purpose, we only consider $w_i > 0, i = 1, \ldots, k$ such that all indicator values are considered in the combined indicator. Based on the above, a Pareto-compliant utility indicator (PCUI) is defined as follows:

Definition 18 (Utility indicator)**.** *Given an utility function $u : \mathbb{R}^k \rightarrow \mathbb{R}$, an indicator vector $\vec{I} \in \mathbb{R}^k$ that assess an approximation set $\mathcal{A}$ and a weight vector $\vec{w} \in \mathbb{R}^k$ such that $w_i > 0, i = 1, \ldots, k$, we denote an utility indicator as $u_{\vec{w}}(\vec{I}(\mathcal{A}))$. If u is also order-preserving as required in Theorem 1, $u_{\vec{w}}(\vec{I}(\mathcal{A}))$ is denoted as Pareto-compliant utility indicator.*

In this work, we focused our attention in two utility functions that are order-preserving, namely, the weighted sum function (WS) and the augmented Tchebycheff function (ATCH) [2], [31]. In the following, we prove that both WS and ATCH are order-preserving functions and, thus, can be employed to define PCUIs.

Definition 19. *The weighted sum (WS) is defined by the following formula:*

$$WS_{\vec{w}}(\vec{x}) = \sum_{i=1}^{k} w_i x_i, \tag{8}$$

where $\vec{x}, \vec{w} \in \mathbb{R}^k$ and $w_i \geq 0, i = 1, \ldots, k$.

Lemma 1. *Given two CIVs $\vec{x}, \vec{y} \in \mathbb{R}^k$ and a weight vector $\vec{w} \in \mathbb{R}^k, w_i > 0, i = 1, \ldots, k$, then if $\vec{x} \succ \vec{y} \Rightarrow WS_{\vec{w}}(\vec{x}) > WS_{\vec{w}}(\vec{y})$.*

Proof. Let's prove this lemma by induction. Let us consider, without loss of generality, that the first component of both CIVs is related to a Pareto-compliant indicator and the rest of components are related to weakly Pareto-compliant indicators, i.e., $x_1 > y_1 \wedge x_i \geq y_i, i = 2, \ldots, k$.

Base case: for $k = 2$, we have $x_1 > y_1 \wedge x_2 \geq y_2$. Then $w_1 x_1 + w_2 x_2 > w_1 y_1 + w_2 y_2$.

Inductive hypothesis: Given $\vec{x}, \vec{y} \in \mathbb{R}^k$, then $\sum_{i=1}^{k} w_i x_i > \sum_{i=1}^{k} w_i y_i$.

Inductive step: We want to prove that $\sum_{i=1}^{k} w_i x_i + w_{k+1} x_{k+1} > \sum_{i=1}^{k} w_i y_i + w_{k+1} y_{k+1}$. Without loss of generality, let us assume that the $(k+1)$ components are related to a weakly Pareto-compliant indicator, then $x_{k+1} \geq y_{k+1}$ and for every $w_{k+1} > 0$ it follows that $w_{k+1} x_{k+1} \geq w_{k+1} y_{k+1}$. From the above statement and the inductive hypothesis, we have the following:

$$\sum_{i=1}^{k} w_i x_i + w_{k+1} x_{k+1} > \sum_{i=1}^{k} w_i y_i + w_{k+1} y_{k+1}$$
$$\text{WS}_{\vec{w}}(\vec{x}) > \text{WS}_{\vec{w}}(\vec{y})$$

Hence, $\vec{x} \succ \vec{y} \Rightarrow \text{WS}_{\vec{w}}(\vec{x}) > \text{WS}_{\vec{w}}(\vec{y})$. □

For the Augmented Tchebycheff function, we assume without loss of generality that $\vec{x} \in \mathbb{R}^k_+$. In consequence, the absolute values in the original definition are not necessary. In fact, by not considering the absolute values, the function is order preserving in the whole $\mathbb{R}^k$.

Definition 20 (Augmented Tchebycheff)**.** *Given $\vec{x}, \vec{w} \in \mathbb{R}^k$ with $w_i \geq 0$, the Augmented Tchebycheff function (ATCH) is defined as follows:*

$$ATCH_{\vec{w}}(\vec{x}) = \max_{i=1,\ldots,k} \{w_i x_i\} + \alpha \sum_{i=1}^{k} x_i \tag{9}$$

Lemma 2. *Given two CIVs $\vec{x}, \vec{y} \in \mathbb{R}^k$ and a weight vector $\vec{w} \in \mathbb{R}^k, w_i > 0, i = 1, \ldots, k$, then if $\vec{x} \succ \vec{y} \Rightarrow ATCH_{\vec{w}}(\vec{x}) > ATCH_{\vec{w}}(\vec{y})$.*

Proof. Let's prove this lemma by induction. Let us consider, without loss of generality, that the first component of both CIVs is related to a Pareto-compliant indicator and the rest of components are related to weakly Pareto-compliant indicators, i.e., $x_1 > y_1 \wedge x_i \geq y_i, i = 2, \ldots, k$.

Base case ($k = 2$): Since $\vec{x} \succ \vec{y}$, it is straightforward to see that $A = \alpha(x_1 + x_2) > \alpha(y_1 + y_2) = B$. Let $i^* = \arg\max_{i=1,2}\{w_i x_i\}$ be the index of the maximum value. Then, we have one of the following two cases:

1) if $i^* = 1$, $w_1x_1 > w_1y_1$ and $w_1x_1 \geq w_2x_2 \geq w_2y_2$, or
2) if $i^* = 2$, $w_2x_2 \geq w_2y_2$ and $w_2x_2 \geq w_1x_1 > w_1y_1$.

From these two cases, it is clear that $\max_{i=1,2}\{w_ix_i\} \geq \max_{i=1,2}\{w_iy_i\}$. However, since $A > B$, it follows: $\vec{x} \succ \vec{y} \Rightarrow \max\{w_1x_1, w_2x_2\} + A > \max\{w_1y_1, w_2y_2\} + B$.

Inductive hypothesis: For $\vec{x}, \vec{y} \in \mathbb{R}^k$, it holds that $A = \alpha\sum_{i=1}^{k} x_i > \alpha\sum_{i=1}^{k} y_i = B$ because $\vec{x} \succ \vec{y}$. Let $i^* = \arg\max_{i=1,\ldots,k}\{w_ix_i\}$ and $j^* = \arg\max_{j=1,\ldots,k}\{w_jy_j\}$ be the indexes related to the maximum values. Due to the construction of $\vec{x}$ and $\vec{y}$, we have one of the following two cases:

1) if $i^* = 1$, $w_1x_1 > w_1y_1$ and $w_1x_1 \geq w_tx_t \geq w_ty_t, t = 2, \ldots, k$, or
2) if $i^* \in \{2, \ldots, k\}$, $w_{i^*}x_{i^*} \geq w_{i^*}y_{i^*}$ and $w_{i^*}x_{i^*} \geq w_1x_1 > w_1y_1$ and $w_{i^*}x_{i^*} \geq w_tx_t \geq w_ty_t, t = 2, \ldots, k \wedge t \neq i^*$.

This implies that $w_{i^*}x_{i^*} \geq w_{j^*}y_{j^*}$. However, since $A > B$, then it is clear that $\vec{x} \succ \vec{y} \Rightarrow w_{i^*}x_{i^*} + A > w_{j^*}y_{j^*} + B$.

Inductive step: We want to prove the following:

$$\vec{x} \succ \vec{y} \Rightarrow$$
$$\max_{i=1,\ldots,k+1}\{w_ix_i\} + \alpha\sum_{i=1}^{k+1} x_i > \max_{i=1,\ldots,k+1}\{w_iy_i\} + \alpha\sum_{i=1}^{k+1} y_i.$$

Without loss of generality, let us assume that the $(k+1)$ components are related to a weaky Pareto-compliant indicator, then $x_{k+1} \geq y_{k+1}$. Based on our inductive hypothesis, we know that $A > B$. Taking the summation upper limits equal to $k+1$, the inequality still holds, thus $A = \alpha\sum_{i=1}^{k+1} x_i > \alpha\sum_{i=1}^{k+1} y_i = B$. Additionally, despite of considering the $k+1$ elements, $\max_{i=1,\ldots,k+1}\{w_ix_i\} \geq \max_{i=1,\ldots,k+1}\{w_iy_i\}$ because one of the above two cases happens. Hence, $\vec{x} \succ \vec{y} \Rightarrow \max_{i=1,\ldots,k+1}\{w_ix_i\} + \alpha\sum_{i=1}^{k+1} x_i > \max_{i=1,\ldots,k+1}\{w_iy_i\} + \alpha\sum_{i=1}^{k+1} y_i$. □

V. Experimental results

In this section, we report the experimental results concerning PCUIs. The main goal is to analyze the preferences imposed by PCUIs using MOPs with different Pareto front geometries. For this purpose, a number of MOEAs, having characteristic distributions, are ranked by the PCUIs of Table III and the individual indicators HV, R2, IGD$^+$, and ϵ^+. Then, we correlate the rankings associated to each indicator in order to determine their relationships.

We employed the Lamé superspheres problems [32] using 2, 3 and 4 objective functions. Lamé problems represent the intersection of Lamé superspheres with the positive $\mathbb{R}^m$-orthant. These test problems require a parameter γ that changes the geometry as follows: if $0 < \gamma < 1$, we obtain a convex geometry, $\gamma = 1$ defines a linear Pareto front, and $\gamma > 1$ is related to concave Pareto fronts. Table IV shows the MOPs employed in the experiments with regard to the geometry of the associated Pareto front. The Pareto fronts of Lamé problems are highly correlated with the simplex defined by a set of convex weight vectors[4]. In order to see the performance of PCUIs in problems having Pareto fronts not correlated with the simplex explained above, we employed the Mirror problems [32] that invert the geometry of the Lamé problems.

TABLE III
PCUIs defined on the basis of the indicators HV, R2, IGD$^+$ and ϵ^+, using the order-preserving utility functions WS and ATCH. For ease of notation, we omit $\vec{w}$ as subindex for the PCUIs. In all cases, $\vec{w} = (0.5, 0.5)$.

UI	(HV, R2)	(HV, IGD$^+$)	(HV, ϵ^+)
WS	WS(HV, R2)	WS(HV, IGD$^+$)	WS(HV, ϵ^+)
ATCH	ATCH(HV, R2)	ATCH(HV, IGD$^+$)	ATCH(HV, ϵ^+)

TABLE IV
MOPs employed in the experiments. Mirror problems invert the geometry of the Lamé superspheres problems [32]. The γ value changes the geometry of the Pareto fronts.

γ	Lamé	Mirror
0.25	Convex	Concave
0.50	Convex	Concave
0.75	Convex	Concave
1.00	Linear	Linear
1.50	Concave	Convex
2.00	Concave	Convex
6.00	Concave	Convex

A. Analysis of preferences

When evaluating Pareto front approximations, quality indicators impose different preferences and, thus, MOEAs are usually ranked in a different way [16]. On the one hand, it is well know that approximation sets having a high concentration of points around the knee of the Pareto front are preferred by the hypervolume [33]. On the other hand, the R2 indicator rewards MOEAs that produce evenly distributed solutions due to the use of the convex weight vectors and the employed scalarizing function [21]. However, currently, it is far from being completely understood what are the preferences of QIs. Moreover, to the authors' best knowledge, there are only two works that aim to empirically establish relationships among QIs, these are the studies of Jiang *et al.* [30] and Liefooghe and Derbel [23]. The latter proposes an interesting correlation analysis, using the Kendall's τ correlation [34], among different QIs based on how they rank different distribution of points.

In furtherance of analyzing the preferences of the proposed PCUIs, we follow a similar methodology to that introduced by Liefooghe and Derbel. Hence. The adopted indicators in our study are HV, R2, IGD$^+$, ϵ^+ and the six PCUIs defined in Table III. We selected nine MOEAs that produce particular distributions of points so that we can determine how PCUIs and the base indicators rank them, using the Lamé and Mirror problems of Table IV for 2, 3 and 4 objective functions. The selected MOEAs can be classified according to their design methology as follows:

[4]A vector $\vec{w} \in \mathbb{R}^m$ is a convex weight vector if and only if $\sum_{i=1}^{m} w_i = 1$ and $w_i \geq 0, i = 1, \ldots, m$.

- Pareto-based MOEAs: NSGA-II [4] and SPEA2 [35].
- Decomposition: MOEA/D [7].
- Indicator-based MOEAs: SMS-EMOA [8], MOMBI2 [9], IGD$^+$-MaOEA [36] and Δ_p-MaOEA[5].
- Analysis of Parallel coordinates: MOVAP [37]

The parameter settings for the considered MOEAs are as follows. For 2, 3 and 4 objective functions, the number n of decision variables are 6,7 and 8, respectively. Moreover, we always use a population size of 120 individuals and 50000 function evaluations as stopping criterion. In all cases, for the simulated binary crossover and the polynomial mutation employed by all the MOEAs, the crossover probability is set to 0.9, the mutation probality is equal to $1/n$, and both the crossover distribution and the mutation distribution indexes are set to 20. MOEA/D, NSGA-III, MOMBI2, WS(HV, R2) and ATCH(HV, R2) employ $N = C_{m-1}^{m+h-1}$ weight vectors, where h is set to 119, 14 and 7 for 2, 3, and 4 objectives, respectively. MOVAP needs a resolution factor for the analysis of the parallel coordinates image that, in case of 2 objectives is set to 3, and for 3 and 4 objective functions, it is equal to 2. Finally, $p = 2$ for Δ_p-MaOEA in every case.

According to the theoretical framework of Sect. IV, the combined indicators are to be maximized. However, originally, the aim is to minimized R2, IGD$^+$ and ϵ^+. In order to avoid this issue, we multiply the values by -1 and, then, the values are normalized, including HV, so that the range of all indicators is $[0, 1]$. For HV, the reference point $\vec{z}_{ref} = (1.1, \ldots, 1.1)$. The R2 indicator uses the Vector Angle Distance Scaling utility function [31]. Regarding the reference sets for IGD$^+$ and ϵ^+, Pareto fronts of size 5000 were generate by enumeration for all cases. Then, these supersets were reduced to 200, 300, and 400 points for 2, 3 and 4 objective functions, respectively. We employed a steady-state selection based on the s-energy indicator [27] in order to reduce the supersets and generate an even distribution of points.

1) First-ranked MOEA: Table V shows the first-ranked MOEAs associated to all the considered indicators for the Lamé problems. Due to the good performance of most of the considered MOEAs in these problems, the preferences are not very different from what it is expected. HV prefers in all the problems SMS-EMOA since it is an HV-based MOEA. R2 always prefers MOEAs that use a set of convex weight vectors, i.e., MOMBI2, MOEA/D and NSGA-III because these approaches produce evenly distributed Pareto fronts. IGD$^+$ and ϵ^+ mostly ranked SMS-EMOA as the best algorithm, although for $\gamma = 1.0$, they select MOEAs based on convex weight vectors. Concerning WS(HV, R2) and ATCH(HV, R2), their preferences are similar. It is interesting to see that for convex problems ($\gamma = 0.25, 0.50, 0.75$), both PCUIs rank SMS-EMOA as the best algoritm. However, for $\gamma = 1.0, 1.5, 2.0, 6.0$, both PCUIs mostly prefer NSGA-III and MOEA/D that produce evently distributed solutions. Hence, this provide certain empircal evidence that the combination of indicators help to compensate the weaknesses of the individual indicators. SMS-EMOA produces well distributed solutions on convex problems but not MOEA/D and NSGA-III because the intersection of the convex weight vectors with the Pareto optimal front are concentrated around the knee of front. However, considering concave problems, MOEA/D and NSGA-III evenly distribute solution, meanwhile SMS-EMOA concentrate solutions around the knee. Regarding the remaining PCUIs based on HV, IGD$^+$ and ϵ^+, they completely prefer SMS-EMOA-like distributions. This can be explained by the high correlation, described in Sect. V-A2, of both IGD$^+$ and ϵ^+ with HV.

Before discussing the preferences on the Mirror problems, it is worth mentioning that MOEAs using a set of convex weight vectors, have a bad performance on these MOPs [38]. In this case, the Pareto fronts are not highly correlated with the simplex defined by the weight vectors, thus, some of them do not intersect the $\mathcal{PF}*$. In contrast, MOEAs not using these vectors, .e.g., MOVAP, SPEA2, IGD$^+$-MaOEA, Δ_p-MaOEA and SMS-EMOA tend to have a better performance. Table VI presents the best-ranked MOEAs for each QIs. Unlike Lamé problems, HV does not always prefers SMS-EMOA in this case. For $\gamma = 1.0$, HV rewards MOEA/D and Δ_p-MaOEA. Moreover, IGD$^+$-MaOEA, that has a similar distribution to that of SMS-EMOA, gets a better HV value in some problems. Regarding R2, there is not a cmplete preference for MOEAs using convex weight vectors. For $\gamma = 0.75, 1.0$ and 1.5, SMS-EMOA-like distributions are rewarded. IGD$^+$ and ϵ^+ exhibit a similar behavior to that of HV, although the latter differs a little bit more. The behavior of WS(HV, R2) and ATCH(HV, R2) is not as stable as in the Lamé problems, where they always preferred well-distributed approximation sets. Now, they mostly reward well-distributed fronts, but they sometimes prefer Pareto fronts having numerous solutions along the boundary, leaving the knee poorly populated. One more time, PCUIs based on IGD$^+$ and ϵ^+ have a very similar behavior. In fact, they differ slightly on linear problems. However, for $\gamma = 0.75, 1.0, 1.5, 2.0$, they prefer well-distributed solutions related to IGD$^+$-MaOEA and Δ_p-MaOEA. For the remaining cases, there a tendency to select SMS-EMOA.

2) Correlation Analysis: In order to statistically analyze the preference of the proposed PCUIs, we followed the metholdology of Liefooghe and Derbel [23]. We employed the two-tailed Kedall's τ test, that is a nonparametric measure of correlation employed with ordinal data. In this case the ordinal data correspond to the ranking of MOEAs produced by each of the considered QIs and PCUIs. A significance level of 0.05 is used is the statistical tests. The underlying idea is to observe the correlation of the base indicators (HV, R2, IGD$^+$ and ϵ^+) and the proposed PCUIs of Table III on the Lamé and Mirror test problems, varying the geometry of the Pareto front and the number of objective functions. Each QIs ranks the nine selected MOEAs, namely, SMS-EMOA, MOMBI2, NSGA-II, NSGA-III, SPEA2, MOVAP, MOEA/D, IGD$^+$-MaOEA and Δ_p-MaOEA. The correlation results are shown using the heatmaps of Figures 3 and 4 for the Lamé and Mirror problems, respetively. From these heatmaps, if $p \geq 0.05$, the white color is used to show that the result is non-significant, i.e., there was not enough statistical evidence to reject $H_0 : \tau = 0$. In the following, we focus on discussing

[5]This is an unpublished MOEA proposed by Falcón-Cardona and Coello based on the design of the IGD$^+$-MaOEA.

TABLE V
MOST PREFERRED PARETO FRONT DISTRIBUTION BY EACH BASE INDICADOR AND PCUI FOR THE LAMÉ SUPERSPHERES PROBLEMS VARYING THE GEOMETRY (γ VALUE) AND THE NUMBER OF OBJECTIVE FUNCTIONS.

γ	Dim.	HV	R2	IGD$^+$	ϵ^+	WS(HV, R2)	ATCH(HV, R2)	WS(HV, IGD$^+$)	ATCH(HV, IGD$^+$)	WS(HV, ϵ^+)	ATCH(HV, ϵ^+)
0.25	2	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	MOMBI2	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	MOMBI2	SMS-EMOA	MOMBI2	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
0.50	2	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	MOMBI2	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	MOMBI2	MOMBI2	IGD$^+$-MaOEA	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
0.75	2	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	MOEA/D	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	MOMBI2	MOMBI2	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	MOMBI2	MOMBI2	SMS-EMOA	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
1.00	2	MOEA/D	MOEA/D	SMS-EMOA	MOEA/D	MOEA/D	MOEA/D	SMS-EMOA	SMS-EMOA	MOEA/D	MOEA/D
	3	SMS-EMOA	NSGA-III	MOMBI2	MOMBI2	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	NSGA-III	MOMBI2	SMS-EMOA	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
1.50	2	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	MOEA/D	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
2.00	2	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	MOEA/D	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
6.00	2	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA

TABLE VI
MOST PREFERRED PARETO FRONT DISTRIBUTION BY EACH BASE INDICADOR AND PCUI FOR THE MIRROR SUPERSPHERES PROBLEMS VARYING THE GEOMETRY (γ VALUE) AND THE NUMBER OF OBJECTIVE FUNCTIONS.

γ	Dim.	HV	R2	IGD$^+$	ϵ^+	WS(HV, R2)	ATCH(HV, R2)	WS(HV, IGD$^+$)	ATCH(HV, IGD$^+$)	WS(HV, ϵ^+)	ATCH(HV, ϵ^+)
0.25	2	SMS-EMOA	MOMBI2	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	MOEA/D	SMS-EMOA	IGD$^+$-MaOEA	MOEA/D	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
0.50	2	SMS-EMOA	MOMBI2	SMS-EMOA	SMS-EMOA	MOMBI2	MOMBI2	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	MOVAP	MOEA/D	MOVAP	MOVAP	SMS-EMOA	MOVAP	MOVAP	MOVAP	MOVAP	MOVAP
	4	IGD$^+$-MaOEA	MOEA/D	IGD$^+$-MaOEA	MOEA/D	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA
0.75	2	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	NSGA-III	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	IGD$^+$-MaOEA	SMS-EMOA	MOVAP	MOVAP	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA	MOVAP	MOVAP
	4	IGD$^+$-MaOEA	MOMBI2	IGD$^+$-MaOEA	SPEA2	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA	IGD$^+$-MaOEA	SPEA2	SPEA2
1.00	2	MOEA/D	NSGA-III	MOEA/D	MOMBI2	NSGA-III	NSGA-III	MOEA/D	MOEA/D	MOMBI2	MOMBI2
	3	Δ_p-MaOEA	SMS-EMOA	Δ_p-MaOEA	Δ_p-MaOEA	MOVAP	SMS-EMOA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA
	4	Δ_p-MaOEA	SMS-EMOA	Δ_p-MaOEA	Δ_p-MaOEA	SPEA2	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA
1.50	2	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	Δ_p-MaOEA	SMS-EMOA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA
	4	IGD$^+$-MaOEA	SMS-EMOA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA
2.00	2	SMS-EMOA	NSGA-III	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	Δ_p-MaOEA	MOEA/D	Δ_p-MaOEA	SMS-EMOA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA	Δ_p-MaOEA
	4	IGD$^+$-MaOEA	MOVAP	Δ_p-MaOEA	Δ_p-MaOEA	MOVAP	Δ_p-MaOEA	Δ_p-MaOEA	IGD$^+$-MaOEA	Δ_p-MaOEA	IGD$^+$-MaOEA
6.00	2	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	3	SMS-EMOA	MOEA/D	SMS-EMOA	MOEA/D	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA
	4	SMS-EMOA	MOEA/D	SMS-EMOA	IGD$^+$-MaOEA	MOVAP	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA	SMS-EMOA

the correlation of the PCUIs.

Considering the relationship between WS(HV, R2) and HV on the Lamé problems, there are some patterns. For 2 objective functions, these indicators are highly correlated for concave problems. However, for convex fronts, the results of the statistical test are not significant, thus, the ranks of WS(HV, R2) and HV are independent. In the three-dimensional case, the indicators tend to be correlated, although in some cases $\tau \in (0.5, 0.75]$. For 4 objective functions, the ranks are independent for problems where $\gamma = 2.0, 6.0$. It is worth emphasizing that for linear fronts, both indicators are highly correlated. On the other hand, regarding WS(HV, R2) and R2 for 2 objectives, there is not statistical evidence that correlate them when $\gamma = 0.25, 0.50, 2.0, 6.0$. This behavior is similar for 3 objectives, whereas for 4 objectives, both indicators are highly correlated. The statistical evidence indicates that WS(HV, R2) and ATCH(HV, R2) are highly correlated in most of the test problems. Regarding the Mirror problems and the correlation between WS(HV, R2) and HV, we observe that they are positively correlated in all three-dimensional problems. However, for convex problems having 2 objective functions, there is not evidence of correlation. This also happens for concave problems in 4 dimensions. In case of the R2 indicator, the correlation tends to be non-significant for highly convex and concave problems.

For Lamé problems with $\gamma = 0.25, 0.50, 0.75, 1.0, 1.5$ in all the considered number of objective functions, ATCH(HV, R2) and HV are positively correlated. However, there is not statistical evidence for highly concave problems when the number of objective functions increases. In contrast, then rankings of ATCH(HV, R2) and R2 are independent for convex problems. Thus, the results with regard to HV and R2 are in some way in conflict. Taking into account the Mirror problems, ATCH(HV, R2) and HV are not as correlated as in the case of Lamé problems. The main issue is related to problems having a high grade of convexity and concavity. These pattern is consistent with the correlation with the R2 for 2 and 4 objective functions.

Regarding the PCUIs based on HV, IGD$^+$ and ϵ^+, we can distinguish a common pattern. The preferences of these four PCUIs are highly correlated with those of the HV. From Figures 3 and 4, we can see that IGD$^+$ and ϵ^+ are positively correlated with HV. Hence, it is expected that the preferences of the proposed PCUIs are aso closely related to HV. Using this information, we can explain why in Tables V and VI the best-ranked MOEAs of the above mentioned PCUIs are very similar to the most preferred MOEA by HV. Additionally, the PCUIs based on the same weakly Pareto-compliant indicator are also very correlated. Hence, this implies that the election of the order-preserving utility functions are not changing the preference regarding nondominated solutions.

VI. Conclusions and Future Work

In this paper, we proposed a novel methodology for the combination of quality indicators in order to produce Pareto-compliant combined indicators. To the authors' best knowledge, this is the first work that proposes such a combination of QIs. For the combination, we need two or more QIs where at least one them is Pareto-compliant and the others are weakly-Pareto compliant. Moreover, an order-preserving function is necessary to combine the indicators value such that the new indicator is Pareto-compliant. We decided to use order-preserving utility functions such as weighted sum and augmented Tchebycheff to show the properties of these new Pareto-Compliant Utility Indicators (PCUIs), combining the hypervolume indicator (the only Pareto-compliant indicator so far) with the indicators R2, IGD^+ and ϵ^+ that are weakly Pareto-compliant. Our experimental results showed that the PCUIs based on IGD^+ and ϵ^+ are highly correlated to the hypervolume indicator, taking into account the preferences of solutions. However, PCUIs based on HV and R2 are the more interesting because the empirical analysis shows that weaknesses of one indicator are cover by the advantages of the other. As part of our future work, we aim to analyze the properties of PCUIs based on HV and R2 in different Pareto front geometries.

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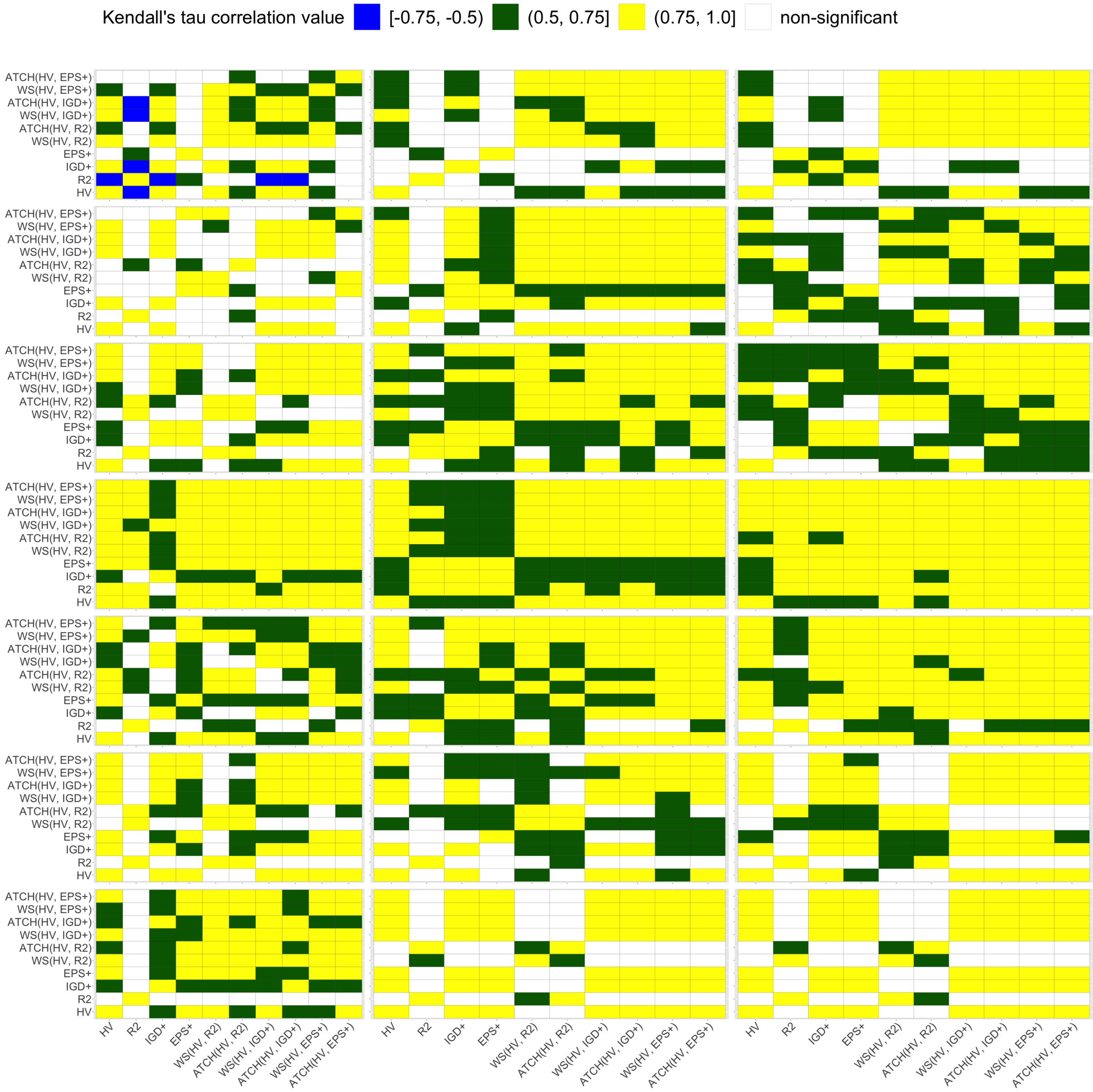

Fig. 3. Heatmap Kendall rank correlation τ for each pair of set quality indicators, each Lamé problem on different dimensions of the objective space.

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Fig. 4. Heatmap Kendall rank correlation τ for each pair of set quality indicators, each Mirror problem on different dimensions of the objective space.

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